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Plimpton 322 Tablet as a Sumerian's Ancient Boundaries Record

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ABSTRACT: One of the most astonishing tablets found in ancient Mesopotamia is the well-known Plimpton 322 Tablet. This tablet is a piece of clay recording fifteen rectangle triangles with integer number sides. It has been abundantly analysed and one of its most recent interpretations is that from Britton *et al.* (2011) who relate the Pythagorean triples with expert work of a scribe. They, also, assume the idea of covering the entire tablet with data including its reverse, which would include up to 23 rows with the corresponding triples. However, the only confirmed data are those of the obverse. Using the Joyce's values for angles W of the corresponding triangles, in this paper, we consider that the triples can be visualized as gnomonic triangles (gnomons and their shadows at midday); then we suggest a new interpretation for the data appearing in the Tablet 322 of the Plimpton's catalogue: they could represent a record of gnomonic locations of "boundary stones" (being W angles Latitudes) and consequently, they could be definitions of specific sites at the time of Sumerian world. The right triangles shown in the tablet could have been observed with a gnomon nearby cities like Terqa, Eshnunna, Akshak, Adab and Nippur in the northern part of Sumer, at the day of equinox around 4,000 years ago. Other cities to the north,

outside this region, would be indicated by the other triples. If we assume that the origin of the Plimpton 322 tablet could be Larsa, a city nearby Uruk, then, we can suggest that a missing part of the broken tablet would include up to 70 mm of data (according to Britton et al.) of an additional column for the side (h) itself and a possible column for the place; at least two more rows: one for a triangle 875:1440:1685 at Latitude of Larsa and another corresponding a triangle 611:1020:1189 for Ur city. These last triples were found with a methodology based on the properties of Pythagorean triples of Plimpton 322 tablet and reported elsewhere: the h side must be a multiple of six.

KEY-WORDS: Gnomonic factor, Sun's altitude, boundary stones

1. Introduction

The clay tablet with the catalogue number 322 in the G. A. Plimpton Collection at Columbia University was discovered in an unknown place in the desert of Iraq and it represents the most well-known tablet that includes a mathematical content. It was scribed in ancient Babylonia (Old Babylonia around 1800 BC) and describes 15 rectangle triangles making it the most advanced mathematical tablet before Greek mathematics. The exact origin of the tablet and its content are unknown, neither the reason why it was written is certain. E. Robson (2002) describes the tablet as an exercise of Pythagorean triples and assumes to be Larsa, an ancient Sumerian city near Uruk, as the place where it was written. Britton et al. (2011) describe the tablet as having a writing surface of 125 mm by 83 mm in the obverse but "... total vertically scored but empty on the lower edge and revers equals ~105 mm (lower edge, 22 mm; reverse, 83 mm)".

Larsa was an important city in Mesopotamia. This Old Babylonian site comes from the Sumerian period. One interesting fact is that the cities in this period were located in the south between Tigris and Euphrates rivers. In Figure 1 it is possible to see some of them. Cities like Ur, Uruk, Larsa, Lagash and Nippur come with the more northern ones: Eshnunna, Akshak and Adab.

There is evidence that the "Pythagorean" theorem relating the squares of the sides with the square of the hypotenuse was known more than a thousand years before Pythagoras' time. The right triangle with integer sides was surely known in

Mesopotamia (Neugebauer, 1969) and Egypt (Magdolen, 2001) with the 3:4:5 triangle as its best representation. Then it is not surprising to find there evidence of other integer number triangles; what is surprising is that such triangles appear in the tablet.

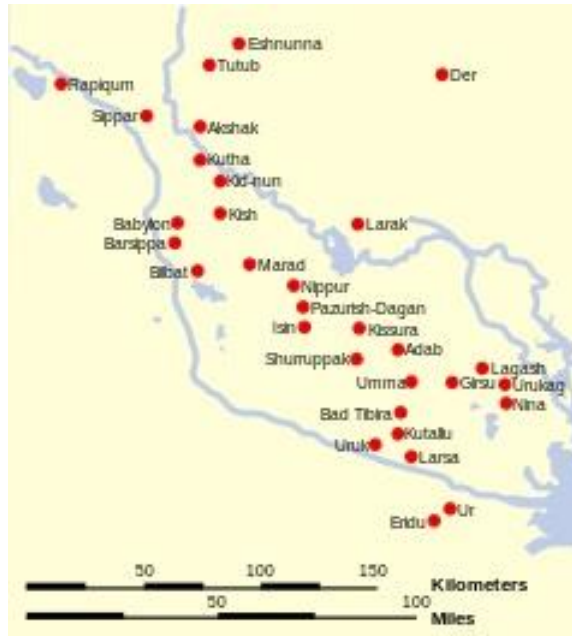


Figure 1 Important cities in Sumerian times.

On one side of the coin, we must consider the possible origin of the knowledge of the triangle itself. R. Perez-Enriquez has proposed that the concept came with the recognition of the gnomon and its shadow as the sides and the ray coming from the Sun as the hypotenuse (2000, 2014). Establishing this way a connection between Time (movement of the Sun along the day and along the year) and mathematics: the *gnomonic factor* (fg) as its best expression. R. Calvino suggests that the *homo sapiens sapiens* got his evolution because he himself became a gnomon (2014). One fact is sure, some important ancient cities were located where the gnomonic factor was an integer: Teotihuacan and Chchen-Itza in Mesoamerica with $fg=1$; Stonehenge in UK having a factor of 3; and Newgrange in Ireland back to 5,000 years ago, with $fg=4$ (Perez-Enriquez, 2014b).

On the other side, we found the Robson hypothesis (Robson, 2002) suggesting the Tablet 322 as a matter of an exercise of a scribe whom systematically found these triangles. No other possible relation with mathematics is given by her.

However, as we will show later the triples selected by the scribe have implicit an angle (W) that decreases steadily.

In this paper we will describe briefly the Plimpton 322 Tablet, introducing the most recent interpretation; then, we are going to work with the triangles using a methodology developed by Eugenio Ley-Koo (2015) in order to find families of Pythagorean triples and factors relating them; from the triangles and the angles represented in the tablet, we go through the ancient world to find sites or cities where at the equinox, such triangles could have been observed. Finally, we come up with a proposal suggesting that the tablet represents triangles associated to sites where the observation of the Sun with a gnomon was done.

2. The Tablet's Content

Relevant descriptions of the Plimpton 322 Tablet can be found easily in the literature. O. Neugebauer described it and gave a coherent interpretation of its four columns. Figure 2 is one of the several pictures from this important tablet. An analysis and translation of the cuneiform text tells us that fifteen triangles with integer sides are inscribed. Columns two and three give the width (w) and the diagonal (d); while first column shows the square of the relation between d and h (see Figure 3). The fourth one is only for ordering, a numeral column.

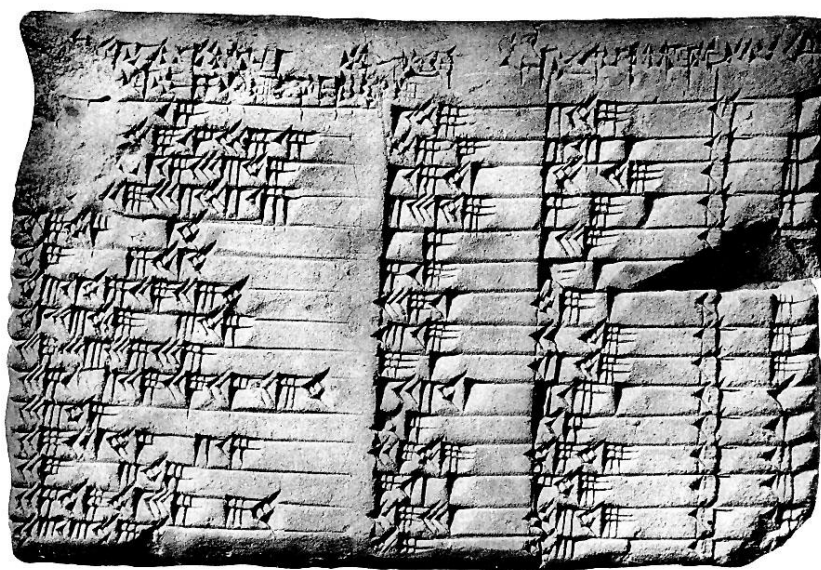


Figure 2 Tablet with number 322 in the Catalogue of the G.A. Plimpton Collection at Columbia University (CDLI Archive).

In order to make accessible the discussion we present through this paper, we must assume that the triples are the following: width: height: diagonal ($w:h:d$) with corresponding angles $W:H:D$. We consider D as 90° . We are taking the names from the columns in the tablet and the name “height” side will become apparent later.

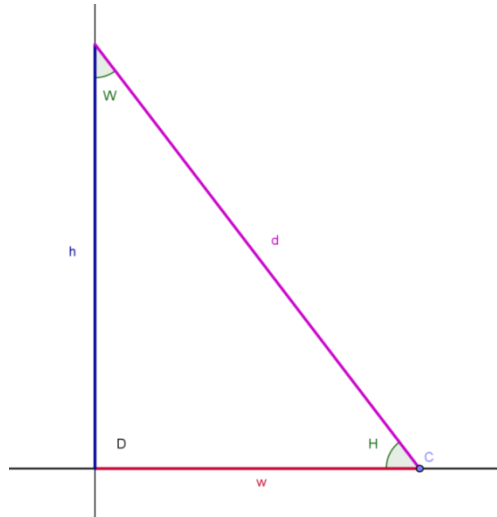


Figure 3 Description of the triangles: width, height and diagonal ($w:h:d$) with corresponding angles $W:H:D$.

According to Britton *et al.*, the tablet contents would be the columns shown in Figure 4; where we have included up to row 25th of their conjecture (partial view of their Table 6). Joyce, at his time, affirms that the first columns is $(d/h)^2$. Translation of the first column head could be read, according to Robson, as “The takiltum-square of the diagonal from which 1 is torn out, so that the short side comes up”.

Table 6 Plimpton 322. Conjectured reconstruction of the contemplated completed tablet

β [I']	δ [O']	$\delta^2 = 1 + \beta^2$ I'	b II'	d III'	N° IV'
sag	si-li-ip-tum	ta-k]i-li-ti si-li-ip - tim ša 1 in]-na-as-ša-hu-ù-ma sag i-li-lu-ù	ib.sib sag	ib.sib si-li-ip-tim	mu.bi.im
59 30	1 24 30	1 59 0 15	1 59	2 49	ki 1
58 27 17 30	1 23 46 2 30	1 56 56 58 14 50 6 15	56 7	1 20 25	ki 2
57 30 45	1 23 6 45	1 55 7 41 15 33 45	1 16 41	1 50 49	ki 3
56 29 4	1 22 24 16	1 53 10 29 32 52 16	3 31 49	5 9 1	ki 4
54 10	1 20 50	1 48 54 1 40	1 5	1 37	ki 5
53 10	1 20 10	1 47 6 41 40	5 19	8 1	ki 6
50 54 40	1 18 41 20	1 43 11 56 28 26 40	38 11	59 1	ki 7
49 56 15	1 18 3 45	1 41 33 45 14 3 45	13 19	20 49	ki 8
48 6	1 16 54	1 38 33 36 36	8 1	12 49	ki 9
45 56 6 40	1 15 33 53 20	1 35 10 2 28 27 24 26 40	1 22	2 16 1	ki 10
45	1 15	1 33 45	45	1 15	ki 11
41 58 30	1 13 13 30	1 29 21 54 2 15	27 59	48 49	ki 12
40 15	1 12 15	1 27 0 3 45	2 41	4 49	ki 13
39 21 20	1 11 45 20	1 25 48 51 35 6 40	29 31	53 49	ki 14
37 20	1 10 40	1 23 13 46 40	28	53	ki 15
36 27 30	1 10 12 30	1 22 9 12 36 15	2 55	5 37	ki 16
32 50 50	1 8 24 10	1 17 58 56 24 1 40	7 53	16 25	ki 17
32	1 8	1 17 4	8	17	ki 18
30 4 53 20	1 7 7 6 40	1 15 4 53 43 54 4 26 40	1 7 41	2 31 1	ki 19
29 15	1 6 45	1 14 15 33 45	39	1 29	ki 20
27 40 30	1 6 4 30	1 12 45 54 20 15	6 9	14 41	ki 21
25	1 5	1 10 25	5	13	ki 22

Figure 4 Conjectured Plimpton 322 of Britton et al. (Table 6); numbers are in sexagesimal notation (Britton 2011).

Relevant to our discussion is to consider the proposal made by Joyce assigning one missing column with the Angle W (he names it angle A after column 1 $((d/h)^2$ that he mentions as $(c/b)^2$) as it is shown in Figure 5.

<u>angle A</u>	<u>$(c/b)^2$</u>
44.76	1:59:00:15
44.25	1:56:56:58:14:50:06:15
43.79	1:55:07:41:15:33:45
43.27	1:53:10:29:32:52:16
42.08	1:48:54:01:40
41.54	1:47:06:41:40
40.32	1:43:11:56:28:26:40
39.77	1:41:33:45:14:03:45
38.72	1:38:33:36:36
37.44	1:35:10:02:28:27:24:26
36.87	1:33:45
35.78	1:31:09:09:25:42:02:15
34.98	1:29:21:54:02:15
33.86	1:27:00:03:45
33.26	1:25:48:51:35:06:40
31.89	1:23:13:46:40

Figure 5 Angles from Plimpton 322 Tablet data proposed by Joyce.
Angle A corresponds to Angle W in this paper (Joyce, 1995).

3. Pythagorean Triangles Identified

The first two columns of the Britton's Table 6, shown above, assumed the value came from a method similar to the one used by Casselman to found the Pythagorean triplet primitives (Casselman, 2003):

“If (A, B, C) is Pythagorean triple, then we can write it as (mw, mh, md) where (w, h, d) is a **primitive** Pythagorean triple - one in which the numbers are relatively prime in pairs. Primitive Pythagorean triples are parameterized by pairs of integers (p, q) satisfying these conditions:

- p and q are both positive;
- p is greater than q ;
- one of them is odd, the other even;
- p and q are relatively prime

“The pair (p, q) gives rise to the triple $(p^2 - q^2, 2pq, p^2 + q^2)$. The pair (p, q) can be easily recovered from (w, h, d) by the formulas $p^2 = (w + d)/2$, $q = h/2p$.

Here are the values of p , q , and m for the triples in the table”

However, Robson considers, using cultural and social arguments, that this approach could not be the one used by Scribes of Mesopotamia. Instead, she supports the interpretation given by Bruins in 1955 as a more feasible one. In such case sides would be found by the following relations

$$\begin{aligned} w' &= w/h = (x - 1/x)/2, \\ h' &= h/h = 1, \\ d' &= d/h = (x + 1/x)/2, \end{aligned} \tag{1}$$

With the integer and it's reciprocal in natural order going down. She proposed that the tablet then represents a "simple mathematical exercise" (Robson, 2002).

Our approach to an interpretation of the tablet is different. As it will be shown in next section, we consider that the triangles could represent a set of special Pythagorean triples, obeying a rule for angles W going from 45° to 31° . But before we continue, and in order to analyse the information of triangles in the tablet, we have follow an alternative pathway. We do not propose that this path was used by Sumerians; we just try to establish a relation between the angles W deductibles from the existing information and the missing sides h .

Assuming that all the triangles in the tablet have a h side that is multiple of 6 (really most of them are multiples of 30) and known that the other sides must be odd; i.e., there must be an odd number between d and h , and, w must be an odd number itself; we found the factors needed in order to establish a direct relation between the W angle to the h side. This approach is reported in a more mathematical paper in preparation (Ley-Koo, 2015).

As we can see in Table 1, the values for these factors allow us to represent the triangles of the Plimpton 322 Tablet with the heights (h) and the angles H and W for each of the fifteen rows of the tablet.

Table 1 Values of l and k for Plimpton 322 Triples						
Num	h	H	W	X	l	k
1	120	45.240	44.760	0.992	0.238	25
2	3456	45.747	44.253	0.974	0.243	685
3	4800	46.213	43.787	0.959	0.248	925
4	13500	46.729	43.271	0.941	0.254	2521
5	72	47.925	42.075	0.903	0.267	13
6	360	48.455	41.545	0.886	0.273	61
7	2700	49.685	40.315	0.849	0.287	421
8	960	50.230	39.770	0.832	0.294	145

9	600	1.282	38.718	0.802	0.308	85
10	6480	52.563	37.437	0.766	0.325	841
11	60	53.130	36.870	0.750	0.333	8
12	2400	55.024	34.976	0.700	0.362	265
13	240	56.145	33.855	0.671	0.381	25
14	2700	56.738	33.262	0.656	0.391	265
15	90	58.109	31.891	0.622	0.417	8.5

Now, we can go to the gnomonic proposal for the set of triples of Plimpton 322 tablet, because we can go from an angle to a triple.

4. Gnomonic Factor and Latitudes of Sites

The behaviour of the triangles inscribed in the Plimpton 322 Tablet could give us some idea about how they were selected, if it is so, and why they correspond to a specific range of triangles: where W angles are about equidistant in the interval going from 32° to 45° . Now, it will become clear why we have selected the name height (h) for the missing side of the triangle.

The observation of the Sun by ancient cultures is well-known because it appears as a god in practically all of them. Egyptian, Mayas, Hindus as examples, include the observation of the Sun. Consequently, the annual and daily movements of our star must have been known precisely. We have proposed, as R. Calvino says, that man was aware of space and time at the moment he stood up on his feet and became himself a gnomon (Calvino, 2014). From this knowledge and from his understanding of the relation between the Sun rays and the shadow of a gnomon, he began to develop the concept of triangle. Yes, a rectangle triangle could be recognized from the *height* of the gnomon and the *width* of the shadow; being the *diagonal* the ray coming from the Sun.

Few years ago, Perez-Enriquez introduced the concept of *gnomonic factor* (fg) from an analysis of the shadows cast by a gnomon in the extreme positions of the Sun at midday (transit) (Perez-Enriquez, 2000). Later, we introduced the concept of *platonic gnomonic factor* in order to differentiate the former from the one obtained at the winter solstice and the moment when the shadow and the gnomon are the same (gnomon day), for the covering the sites of Egyptian culture (Perez-Enriquez, 2013). Today, we recognize in the angle W implied by the triangles of the Plimpton

322 Tablet as a possible specification of Latitudes of sites in antiquity. We can remember the words of L. Cottrell when he talks about conflicts between ancient cities: “The border markers or “stelas”, were put to indicate the limits belonging to each city...”, (Cottrell, 1962). Moreover, as Kramer says when talking about Mesilim, the king of Kish: “[He] proceeded to arbitrate the controversy (between cities Lagash and Umma) by measuring off the boundary line between the cities in accordance with what was given out to be an oracle of Sataran, a deity in charge of the settling of complaints...” (Kramer 1987) 54. We can make a question about the methodology used for defining the border line. Yes, there is a vast literature recovering the inscriptions contained in the “boundary stones” like the one sold by A. Michaux to the Bibliothèque Nationale in Paris, in which it can be read:

“The army of heaven will water us with vinegar in order to lavish on us the right remedies to affect our healing”. (Kramer) 7

But few words appear telling us how Enki the Sumerian god of wisdom, fixed the borders and seated the boundary stones while putting the Sun-god, Utu, over the entire universe.

In effect, if we consider the values of the angles in Table 2 as values of Latitude (W) and height of the Sun above the horizon (H), we would find that all values are in a range covering the Mesopotamian and the Assyrian regions going up until modern Armenia and Caucasus. These data could represent specific sites for “boundary stones” where in the equinox the Sun was observed to produce each of those special gnomonic triangles: “Pythagorean” triangles with integer numbers as their sides. We have searched for significant sites or cities of Sumerian time that were located nearby these latitudes. The results of this search are shown in Table 3.

Furthermore, following the map of Mesopotamia and Assyria it is easy to look for other important places in the region: Babylon, Larsa, Lagash, Ur. One would expect that those important sites would be included. Babylon could be out of scope because it was not already as important as Larsa or Lagash 4,500 years ago but it would be located between 14th and 15th rows. For the others, one would look forward to find them.

Table 3 Cities at Latitudes Appearing in Plimpton 322 Tablet					
Num	Lat ¹	Shadow ²	Height ³	City ⁴	Alt. City
1	44.760	9.92	45.240	Praskoveya	
2	44.253	9.74	45.747	Obilnoye	Gorkaya Balka
3	43.787	9.59	46.213	Uchebnyy	
4	43.271	9.41	46.729	Sredniy Uruk	Kardzhin
5	42.075	9.03	47.925	Tsinagari	
6	41.545	8.86	48.455	Rustavi, Georgia	Tetritsq'aro
7	40.315	8.49	49.685	Kaputan, Armenia	
8	39.770	8.32	50.230	Arnak, Armenia	
9	38.718	8.02	51.282	Kalavanis	Korkut, Turkey
10	37.437	7.66	52.563	Uludere	
11	36.870	7.50	53.130	Duhak	
12	34.976	7.00	55.024	Laqlaq	
13	33.855	6.71	56.145	Eshnunna	
14	33.262	6.56	56.738	Akshak	
15	31.891	6.22	58.109	Adab	Nippur
1 Latitudes from tablet Plimpton 322 2 For a 10 units gnomon 3 Sun above the horizon at Equinox ($90^\circ - \text{Lat}$) 4 Only 11 to 15 are ancient cities of Sumerian time (Google Earth, 2014)					

Based on the idea expressed by Kramer in the sense that markers were located to define borders between cities and kingdoms (Kramer, 1962) 53, we have explored with the aid of our methodology, the location of these sites: with the corresponding Latitudes of the site we calculate l and k using a proposed value for h . Results are shown in Table 4. There, it is possible to identify the triangles corresponding to these sites. Definitely, Babylon had to be included because there is a very big gap between adjacent latitudes (33.26° to 31.89°). For the cases of Lagash we could not find any triple. For Ur the triple is 611:1020:1189 confirming its presence.

Table 4 Proposed Missing Rows of the Plimpton 322 Tablet						
W	D	Row	H	H	W	City
1235	2293	14 bis	1932	57.412	32.588	Babylon
875	1685	16	1440	58.716	31.284	Larsa
611	1189	17	1020	59.078	30.922	Ur

The case of Larsa, the suggested place for manufacture of the tablet, must be commented separately. We have found several triples for Larsa; the one shown in the Table 4, (875:1440:1685), appears conveniently located. Our Pythagorean triples seem to confirm that the triples in the tablet could represent triangles where the side h is a multiple of six. Then, the triples for Babylon, Larsa and Ur can be located easily in rows 14 bis, 16 and 17. If we remember that the Plimpton 322 tablet have been recovered broken, we can suggest that the tablet was made by a Scribe at Larsa but it had at least two missing rows.

5. Final Comments

We have used a methodology for the analysis of the Plimpton 322 Tablet assuming that the Pythagorean triple present can be completed with a side h for height (multiple of six) and a corresponding angle W , related to the width w , as a Latitude. We suggest that all h sides having values multiple of six and sides w and d separated are odd numbers; also, we used the fact that W angles are almost evenly distributed between 32° and 45° .

Later, we use this methodology and bonded it with the interpretation of a rectangle triangle as a representation of a gnomon and its shadow (a gnomonic triangle); furthermore, we suggest to visualize W angles as the Latitude of a Place where, at the equinox, the corresponding triangle was observed. The interpretation of the Plimpton 322 Tablet of Pythagorean triples defining possible markers in a route between regions in Sumerian time's falls immediately. Figure 7 shows the Middle East region with the sites identified with red points.



Figure 7 Possible sites of observation of Pythagorean triples as triangles formed by a gnomon and the shadow at equinoxes.

Evidences of the use of “boundary stones” are present in the literature as ways to establish borders between cities and kingdoms. For example, the conflict between Lagash and Umma was because a broken border agreement (Kramer, 1962) 53. Evidence of commercial routes between regions is present in cuneiform tablet, also. Figure 8 shows how are distributed the triangles given when a gnomon of one unit is used.

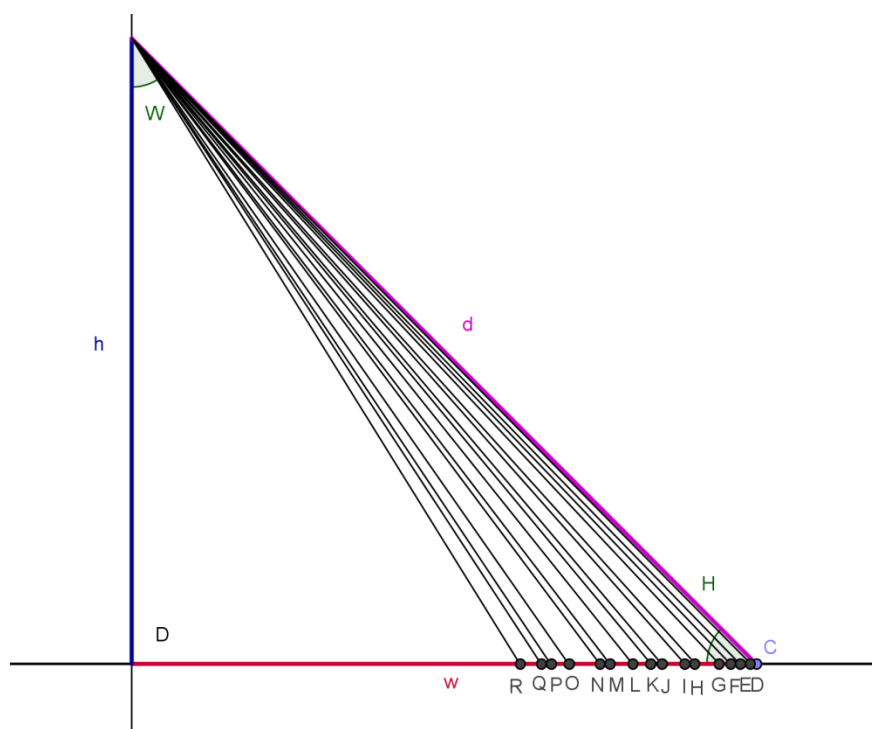


Figure 8 Plimpton 322's Triangles formed by a common gnomon of one unit height and shadows identified from D to R due to the solar altitudes at the equinox.

With these ideas in mind, we have found that Babylon, Larsa and Ur are located at latitudes where Pythagorean triples similar to Plimpton 322 tablet's ones, can be calculated. In consequence, we propose that the multi-mentioned tablet could have had at least two more rows and a column with values of h side or with a name of a city.

One last issue that we can argue is: Why do the tablet begins at 32° and does not cover a range from 30 to 45 degrees? This range would be expected in the exercise interpretation of the Tablet. However, as we are showing in Figure 7, the Persian Gulf at Sumerian times reached Ur at 31° Latitude, Only the city of Eridu goes beyond that limit.

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